Eur. Phys. J. D **6**, 533–539 (1999)

THE EUROPEAN PHYSICAL JOURNAL D
EDP Sciences

Analytical solutions for the energy distribution of charged particles in a weak electric field

G.G.M. Coppa[a] and A. D'Angola

Istituto Nazionale per la Fisica della Materia and Dipartimento di Energetica, Politecnico di Torino, Corso Duca degli Abruzzi, 24 - 10129 Torino, Italy

Received: 28 July 1998

Abstract. The paper deals with the stationary distribution of charged particles moving in a material medium, having scattering and absorption properties, in which a uniform electric field is present. The purpose of the work is finding analytical solutions in simplified but physically significant situations and comparing different approximations based on a spherical-harmonics expansion of the velocity distribution.

PACS. 52.25.Dg Plasma kinetic equations – 51.10.+y Kinetic and transport theory of gases – 05.20.Dd Kinetic theory

1 Introduction

The calculation of the energy distribution of charged particles moving in a material medium under the effect of an external electric field is an important topic in many fields, such as partially ionized plasmas and semiconductor physics. As the interaction between charged particles is generally negligible, such phenomena are well described by a linear Boltzmann equation. A classic technique of solution is based upon the expansion of the angular dependence of the distribution function (of the charged particles) in terms of spherical harmonics (P_N method). Although for strong fields many terms of the expansion must be considered [1–10], for weak electric fields, the P_1 approximation (*i.e.*, by retaining only the first two terms of the expansion) provides sufficiently accurate results [11–16]. Notwithstanding the linearity of the equations, in situations of practical interest the interaction between charged and neutral particles is so complicated that the P_1 equations can be solved only numerically. In the present paper, simplified cases are considered in which an analytical solution to the P_1 model can be found. From a practical point of view, these solutions are important as they provide analytical benchmarks for the numerical codes. Moreover, analytical solutions may represent the starting point of new numerical techniques. In fact, similar procedures are employed in the field of neutron transport, when the problem of the slowing down of fission neutrons is considered.

Although the Boltzmann equation for charged particles is basically different from the one for neutrons, due to the presence of an additional term accounting for the electric field, the two equations present important similarities [17]. In the present paper, the classic problem of determining the energy spectrum of neutral particles in a non-absorbing medium [18] is generalized to include the effect of the electric field. The analytical solution is found by using the P_1 expansion of the angular dependence of the particle density. Within this technique, different approximations are possible, ranging from the simple Fermi approximation to the rigorous calculation of the moments of the collision integral. The analytical solution for each of them is calculated in the following; results are presented showing the validity of the approximations employed.

[a] e-mail: ggmcoppa@polito.it

2 The linear Boltzmann equation for charged particles

The velocity distribution of particles of mass m and charge e in an infinite, homogeneous medium with a constant, uniform electric field $\mathbf{E}$ is governed by the Boltzmann equation [19]

$$\frac{e\mathbf{E}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}}+\mathcal{N}\sigma(v)vf=Q[f]+\frac{S(v)}{4\pi}\,, \qquad (1)$$

being $\sigma(v)$ the total cross-section (including scattering and absorption) and $\mathcal{N}$ the atomic density, while $Q[f]$ and $S(v)$ represent the scattering and external sources, respectively. The scattering source can be written as

$$Q[f]=\int \mathcal{N}\sigma_s(\mathbf{v}'\to\mathbf{v})v'f(\mathbf{v}')\mathrm{d}\mathbf{v}'\,, \qquad (2)$$

being $\sigma_s(\mathbf{v}'\to\mathbf{v})$ the differential scattering cross-section. In the following, all quantities will be expressed as functions of the particle kinetic energy, ϵ, and direction, $\mathbf{\Omega}$.

Moreover, the equations will be written in terms of the angular flux, $\varphi(\epsilon, \mathbf{\Omega})$, defined as

$$\varphi(\epsilon, \mathbf{\Omega}) = \frac{2\epsilon}{m^2} f(\mathbf{v}) \,, \tag{3}$$

in such a way that the following relationship

$$v f(\mathbf{v}) \mathrm{d}\mathbf{v} = \varphi(\epsilon, \mathbf{\Omega}) \mathrm{d}\epsilon \, \mathrm{d}\mathbf{\Omega} \tag{4}$$

holds. Similarly, a new cross-section, $\sigma_s(\epsilon' \to \epsilon, \mathbf{\Omega}' \cdot \mathbf{\Omega})$, and new normalized sources in the $(\epsilon, \mathbf{\Omega})$ domain, $S_\epsilon(\epsilon)$ and $Q_\epsilon[\varphi]$, are introduced, such that

$$\sigma_s(\mathbf{v}' \to \mathbf{v}) \mathrm{d}\mathbf{v} = \sigma_s(\epsilon' \to \epsilon, \mathbf{\Omega}' \cdot \mathbf{\Omega}) \mathrm{d}\epsilon \, \mathrm{d}\mathbf{\Omega} \,,$$

$$S(v) \mathrm{d}\mathbf{v} = \mathcal{N} S_\epsilon(\epsilon) \mathrm{d}\epsilon \, \mathrm{d}\mathbf{\Omega} \,,$$

$$Q[f] \mathrm{d}\mathbf{v} = \mathcal{N} Q_\epsilon[\varphi] \mathrm{d}\epsilon \, \mathrm{d}\mathbf{\Omega} \,. \tag{5}$$

Consequently, equation (1) can be rewritten as

$$\tfrac{1}{2} v e \widehat{\mathbf{E}} \cdot \frac{\partial}{\partial \mathbf{v}} \left(\frac{\varphi}{\epsilon} \right) + \sigma(\epsilon) \varphi(\epsilon, \mathbf{\Omega}) = Q_\epsilon[\varphi] + \frac{S_\epsilon(\epsilon)}{4\pi} \,, \tag{6}$$

being $\widehat{\mathbf{E}} = \mathbf{E}/\mathcal{N}$ the reduced electric field. The scattering source, $Q_\epsilon[\varphi]$, can be expressed in terms of the scattering cross-section and of the angular flux, as

$$Q_\epsilon[\varphi] = \int \mathrm{d}\epsilon' \oint \mathrm{d}\mathbf{\Omega}' \sigma_s(\epsilon' \to \epsilon, \mathbf{\Omega}' \cdot \mathbf{\Omega}) \varphi(\epsilon', \mathbf{\Omega}') \,. \tag{7}$$

In the following, the external source will be considered in the form

$$S_\epsilon(\epsilon) = \delta(\epsilon - \epsilon_0) \,. \tag{8}$$

Therefore, the corresponding solution for φ is the Green function representing the particle flux generated by a source that injects particles all having energy ϵ_0. From this solution, the flux generated by a source of arbitrary spectrum can be readily evaluated.

3 The P_1 approximation

An approximate solution to equations (6, 7) can be calculated by means of the P_1 method [20]. According to this technique, the angular dependence of the flux $\varphi(\epsilon, \mathbf{\Omega})$ is approximated as

$$\varphi(\epsilon, \mathbf{\Omega}) = \frac{\Phi(\epsilon) + 3\mathbf{\Omega} \cdot \mathbf{J}(\epsilon)}{4\pi} \,, \tag{9}$$

being $\Phi(\epsilon)$ and $\mathbf{J}(\epsilon)$ the total flux and the current, respectively, of the particles having energy ϵ. To justify such approximation, it can be observed that, if the electric field were absent, the angular flux would be isotropic and exactly equal to $\Phi(\epsilon)/(4\pi)$. If the electric field is weak (a quantitative condition for that is discussed later), a perturbative solution of the Boltzmann equation shows that a linear term in $\mathbf{\Omega}$ must be added to the angular flux and the expression (9) is obtained; it represents the simplest expression for $\varphi(\epsilon, \mathbf{\Omega})$ when the dependence on $\mathbf{\Omega}$ is to be taken into account. By inserting equation (9) into the transport equation (6) and integrating over $\mathbf{\Omega}$, one obtains

$$e\widehat{\mathbf{E}} \cdot \frac{\mathrm{d}\mathbf{J}}{\mathrm{d}\epsilon} + \sigma \Phi = \int \sigma_s(\epsilon') \Phi(\epsilon') P(\epsilon' \to \epsilon) \mathrm{d}\epsilon' + \delta(\epsilon - \epsilon_0), \tag{10}$$

being $\sigma_s(\epsilon') = \int \mathrm{d}\epsilon \oint \mathrm{d}\mathbf{\Omega} \sigma_s(\epsilon' \to \epsilon, \mathbf{\Omega}' \cdot \mathbf{\Omega})$ the total scattering cross-section at energy ϵ' and having introduced the function $P(\epsilon' \to \epsilon)$, defined as

$$P(\epsilon' \to \epsilon) = \frac{1}{\sigma_s(\epsilon')} \oint \sigma_s(\epsilon' \to \epsilon, \mathbf{\Omega}' \cdot \mathbf{\Omega}) \mathrm{d}\mathbf{\Omega} \,, \tag{11}$$

which represents the probability for the particle energy to change from ϵ' to ϵ as a consequence of a single scattering. To obtain equation (10), use has been made of the formula

$$\frac{\partial}{\partial \mathbf{v}} \left(A(v) + \mathbf{B}(v) \cdot \mathbf{\Omega} \right) = \frac{\mathrm{d}A}{\mathrm{d}v} \mathbf{\Omega} + \tfrac{1}{v} \mathbf{B} + \mathbf{\Omega}\mathbf{\Omega} \cdot v \frac{\mathrm{d}}{\mathrm{d}v} \left(\tfrac{1}{v} \mathbf{B} \right) \,. \tag{12}$$

Equation (10) is sufficient to determine the particle energy distribution only if $\widehat{E} = 0$. If electric field is present, a second equation, relating Φ and $\mathbf{J}$, is necessary. To this purpose, the transport equation (6) is integrated over $\mathbf{\Omega}$ after multiplying it by $\mathbf{\Omega}$ itself. One obtains

$$\tfrac{1}{3} e \widehat{\mathbf{E}} \epsilon \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(\frac{\Phi}{\epsilon} \right) + \sigma \mathbf{J} = \int \sigma_s(\epsilon') \mathbf{J}(\epsilon') \mu_0(\epsilon' \to \epsilon) P(\epsilon' \to \epsilon) \mathrm{d}\epsilon' \,, \tag{13}$$

being $\mu_0(\epsilon' \to \epsilon)$ the cosine of the deflection angle for the transition $\epsilon' \to \epsilon$, defined as

$$\mu_0(\epsilon' \to \epsilon) = \frac{1}{\sigma_s(\epsilon') P(\epsilon' \to \epsilon)} \oint \mathbf{\Omega}' \cdot \mathbf{\Omega} \sigma_s(\epsilon' \to \epsilon, \mathbf{\Omega}' \cdot \mathbf{\Omega}) \mathrm{d}\mathbf{\Omega} \,. \tag{14}$$

As the electric field and the particle current are parallel (*i.e.*, $\widehat{\mathbf{E}} = \widehat{E} \mathbf{J}/J$), equation (13) is not a real vector equation, as it simply connects $\widehat{E}$ and J.

In the following, the scattering process between a charged particle and an atom of mass M is assumed to be elastic and isotropic in the reference frame of the centre of mass. In this case, the transition probability $P(\epsilon' \to \epsilon)$ is given by [20]

$$P(\epsilon' \to \epsilon) = \begin{cases} \dfrac{1}{(1-\alpha)\epsilon'} \,, & \text{for} \quad \alpha\epsilon' \le \epsilon \le \epsilon' \,, \\ 0, & \text{otherwise} \,, \end{cases} \tag{15}$$

being $\alpha = (\frac{M-m}{M+m})^2$, and the cosine of the deflection angle can be written as

$$\mu_0(\epsilon' \to \epsilon) = \frac{1}{1 - \sqrt{\alpha}} \left(\sqrt{\frac{\epsilon}{\epsilon'}} - \sqrt{\frac{\alpha\epsilon'}{\epsilon}} \right) \,. \tag{16}$$

Moreover, as the purpose of the present work is to obtain analytical reference solutions for the charged-particle energy distribution, a simplified dependence of the cross-sections on energy is assumed, by considering a $1/\epsilon$ behaviour of σ_s and σ:

$$\sigma_s(\epsilon) = \frac{\Upsilon_s}{\epsilon}\ ,\ \sigma(\epsilon) = \frac{\Upsilon}{\epsilon}\ . \tag{17}$$

It turns out useful to rewrite equations (10) and (13) in terms of the lethargy variable $u = \log(\epsilon_0/\epsilon)$ and of the quantities $\mathfrak{F}(u) = \epsilon(u)\sigma_s(\epsilon(u))\Phi(\epsilon(u))$, $\mathfrak{J}(u) = -\epsilon(u)\sigma_s(\epsilon(u))J(\epsilon(u))$, which represent the first moments of the scattering density $\sigma_s(\varepsilon)\varphi(\varepsilon,\mathbf{\Omega})$; the following system of equations is obtained:

$$\begin{cases} \dfrac{e\widehat{E}}{\Upsilon_s}\dfrac{\mathrm{d}\mathfrak{J}}{\mathrm{d}u} + \dfrac{\Upsilon}{\Upsilon_s}\mathfrak{F}(u) = \displaystyle\int_{u-\Delta u}^{u} \frac{\exp(u'-u)}{1-\alpha}\mathfrak{F}(u')\mathrm{d}u' + \delta(u)\,, \\ \dfrac{e\widehat{E}}{3\Upsilon_s}\left(\mathfrak{F} + \dfrac{\mathrm{d}\mathfrak{F}}{\mathrm{d}u}\right) + \dfrac{\Upsilon}{\Upsilon_s}\mathfrak{J}(u) = \displaystyle\int_{u-\Delta u}^{u} \frac{1}{(1-\alpha)(1-\sqrt{\alpha})} \\ \quad \times \left\{\exp\left(\tfrac{3}{2}(u'-u)\right) - \sqrt{\alpha}\exp\left(\tfrac{1}{2}(u'-u)\right)\right\}\mathfrak{J}(u')\mathrm{d}u', \end{cases} \tag{18}$$

where $\Delta u = -\log\alpha$ represents the maximum increase of lethargy due to a single scattering event.

4 Approximate solution to the P_1 model

The quantity $\mathfrak{F}(u)$ represents the scattering density per unit of lethargy; more precisely, $\mathfrak{F}(u)\mathrm{d}u$ is the mean number of scattering events, per unit of time, occurring to each atom of the material medium due to charged particles having lethargy in $(u, u+\mathrm{d}u)$. If $\widehat{E} = 0$, the density $\mathfrak{F}(u)$ can be determined from the first equation (18); its solution, known as Placzek function [18], can be calculated analytically. If the electric field is non-vanishing, also the second equation (18) must be taken into account; in fact, its solution provides the necessary relationship between the "current" $\mathfrak{J}$ and the scattering density $\mathfrak{F}$. However, if the electric field is weak enough, an approximate solution to this equation can be used. The simplest approximation is obtained by assuming $\mathfrak{J}(u') \simeq \mathfrak{J}(u)$ in the integral appearing in the r.h.s. of the second equation (18); in such a way, the integral is approximated simply as $\langle\mu_0\rangle\,\mathfrak{J}(u)$, where $\langle\mu_0\rangle = \frac{2m}{3M}$ represents the mean value of the cosine of the deflection angle [20], and the equation can be solved with respect to $\mathfrak{J}$, obtaining

$$\mathfrak{J}(u) = -\frac{e\widehat{E}}{3\left(\Upsilon - \langle\mu_0\rangle\,\Upsilon_s\right)}\left(\mathfrak{F} + \frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}u}\right)\ . \tag{19}$$

Finally, by inserting equation (19) into the first equation (18), the following integro-differential equation for $\mathfrak{F}$ is obtained:

$$\zeta\left(\frac{\mathrm{d}^2\mathfrak{F}}{\mathrm{d}u^2} + \frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}u}\right) - \frac{1}{\gamma}\mathfrak{F} + \int_{u-\Delta u}^{u}\frac{\exp(u'-u)}{1-\alpha}\mathfrak{F}(u')\mathrm{d}u' + \delta(u) = 0\ , \tag{20}$$

having defined the constant ζ and γ as

$$\zeta = \frac{\left(e\widehat{E}\right)^2}{3\Upsilon_s\left(\Upsilon - \langle\mu_0\rangle\,\Upsilon_s\right)}\ ,\ \gamma = \frac{\Upsilon_s}{\Upsilon}\ . \tag{21}$$

If $\gamma < 1$ (*i.e.*, if absorption is present), the appropriate boundary conditions for equation (20) are

$$\lim_{u\to\pm\infty}\mathfrak{F}(u) = 0\ . \tag{22}$$

In fact, by integrating the transport equation (6) over the $(\mathbf{\Omega}, \epsilon)$ domain, one can readily verify that the total number of scattering events, $\int_{-\infty}^{+\infty}\mathfrak{F}(u)\mathrm{d}u$, can be evaluated as $\gamma/(1-\gamma)$. Being $\mathfrak{F}(u)$ always non-negative, the condition (22) is required. The important case $\gamma = 1$ (*i.e.*, $\sigma_s = \sigma$) will be considered as a limit situation when $\gamma \to 1^-$.

It must be noticed that a qualitative but physically significant solution of equation (20) can be obtained using the so-called Fermi approximation [20], which consists in calculating the integral term by assuming $\mathfrak{F}(u') \simeq \mathfrak{F}(u) + (u'-u)\mathrm{d}\mathfrak{F}/\mathrm{d}u$. Using this procedure, equation (20) becomes a simple Fokker-Planck differential equation:

$$\zeta\frac{\mathrm{d}^2\mathfrak{F}}{\mathrm{d}u^2} + (\zeta - \langle\Delta u\rangle)\frac{\mathrm{d}\mathfrak{F}}{\mathrm{d}u} - \left(\frac{1}{\gamma} - 1\right)\mathfrak{F} + \delta(u) = 0\ , \tag{23}$$

being $\langle\Delta u\rangle = 1 + \frac{\alpha}{1-\alpha}\log\alpha$ the mean increase in lethargy further to a scattering event [20]. The drift coefficient appearing in equation (23), $\langle\Delta u\rangle - \zeta$, is given by the difference between the mean increase in lethargy per scattering and the quantity ζ, whose meaning is evidently the decrease of lethargy caused by the electric field between two successive collisions. The roots λ_1, λ_2 of the characteristic equation associated with equation (23), given by

$$\lambda_{1,2} = \frac{1}{2\zeta}\left\{\langle\Delta u\rangle - \zeta \pm \left[(\langle\Delta u\rangle - \zeta)^2 + 4\zeta\frac{1-\gamma}{\gamma}\right]^{\frac{1}{2}}\right\}, \tag{24}$$

are always real; moreover, they have opposite sign ($\lambda_1 > 0$, $\lambda_2 < 0$) if $\zeta < \langle\Delta u\rangle$. When ζ is greater than $\langle\Delta u\rangle$, the electric field is so strong that the energy gain of a particle between two successive scattering events is larger than the loss due to a single scattering. In such situation, the average energy of each particle increases ceaselessly. As the P_1 approximation requires a weak electric field, the condition $\zeta \ll \langle\Delta u\rangle$ will be considered always satisfied in the following. In this case, the solution of equation (23) is

$$\mathfrak{F}(u) = \frac{1}{\zeta\left(\lambda_1 - \lambda_2\right)} \times \begin{cases} \exp(\lambda_1 u), & \text{for} \quad u < 0\ , \\ \exp(\lambda_2 u), & \text{for} \quad u > 0\ . \end{cases} \tag{25}$$

The exact solution of equation (20) can be calculated by resorting to the Fourier transform technique. Having defined the Fourier transform of a generic function $f(u)$ as $\widetilde{f}(\omega) = \int_{-\infty}^{+\infty} f(u)\exp(-i\omega u)\mathrm{d}u$, from equation (20) one obtains

$$\widetilde{\mathfrak{F}}(\omega) = \frac{1}{\mathfrak{D}_0(\omega)}\ , \tag{26}$$

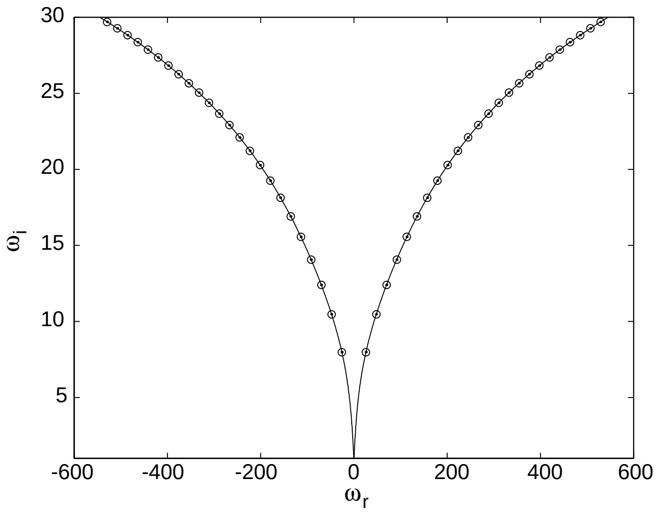

Fig. 1. Location of the zeroes of $\mathfrak{D}_0(\omega)$, for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 10^{-4}$ (circles). The zeroes stand on a curve (full line), deduced in the appendix.

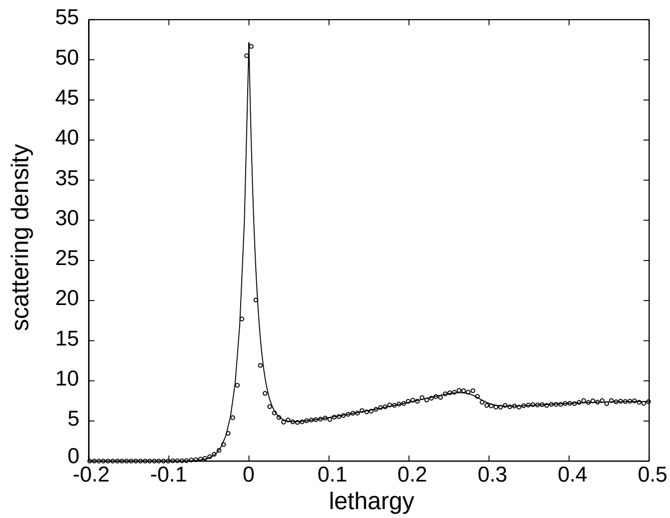

Fig. 2. Scattering density obtained by solving equation (20) (full line) and by Monte Carlo simulation (circles), for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 10^{-4}$.

being

$$\mathfrak{D}_0(\omega) = \frac{1}{\gamma} - i\zeta\omega(1 + i\omega) - \frac{1 - \alpha\exp(-i\omega\Delta u)}{(1-\alpha)(1+i\omega)}\,. \quad (27)$$

The inverse transform,

$$\mathfrak{F}(u) = \int_{-\infty}^{+\infty} \frac{1}{2\pi\mathfrak{D}_0(\omega)} \exp(i\omega u)\mathrm{d}\omega\,, \quad (28)$$

can be evaluated by using the residue theorem. To this purpose, the zeroes of $\mathfrak{D}_0(\omega)$ must be calculated in complex domain, assuming $\omega = \omega_r + i\omega_i$. The equation $\mathfrak{D}_0(\omega) = 0$ has an infinite number of zeroes, Ω_k; two of them, Ω_0 and Ω_1, are purely imaginary and can be approximated as $i\lambda_2$ and $i\lambda_1$, being λ_1, λ_2 the roots of the characteristic equation for the Fermi solution. The remaining zeroes have a non-vanishing real part and a positive imaginary part greater than λ_2. A typical location of the Ω_k's, for $k > 1$, is shown in Figure 1 for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 10^{-4}$. The zeroes in the complex plane stand on a curve, which can be determined analytically, as shown in the appendix. Although the Ω_k can only be determined numerically, the following asymptotic formula holds:

$$\Omega_k \sim \pm y_k + i\left(1 + \frac{1}{\Delta u}\log\left\{\frac{1-\alpha}{\gamma} y_k\left[1 + \zeta\gamma y_k^2\right]\right\}\right), \quad (29)$$

being $y_k = \frac{1}{\Delta u}(\frac{\pi}{2} + 2k\pi)$. Details about the deduction of formula (29) are reported in the appendix.

In order to evaluate $\mathfrak{F}(u)$, the knowledge of the asymptotic behaviour of $\mathfrak{D}_0(\omega)$ for $\omega \to \infty$ is necessary to choose the correct integration path in the ω plane. From equation (27), one can observe that $\mathfrak{D}_0(\omega) \sim f(\omega)\exp(\omega_i\Delta u)$ for $\omega_i \to +\infty$ (being $f(\omega)$ non-diverging for $\omega_i \to +\infty$) and that $\mathfrak{D}_0(\omega) \sim \zeta\omega^2$ for $\omega_i \to -\infty$. Consequently

$$\lim_{\omega_i \to +\infty} \frac{1}{\mathfrak{D}_0(\omega)} \exp(i\omega u) = 0, \quad \text{for} \quad u > -\Delta u\,, \quad (30)$$

while

$$\lim_{\omega_i \to -\infty} \frac{1}{\mathfrak{D}_0(\omega)} \exp(i\omega u) = 0, \quad \text{for} \quad u < 0\,. \quad (31)$$

For that reason, the integral in equation (28) can be evaluated as

$$\mathfrak{F}(u) = \begin{cases} i\sum_{k>0} \operatorname*{Res}_{\Omega_k}\left(\frac{1}{\mathfrak{D}_0(\omega)}\right)\exp(i\Omega_k u), & \text{for } u > -\Delta u, \\ -i\operatorname*{Res}_{\Omega_0}\left(\frac{1}{\mathfrak{D}_0(\omega)}\right)\exp(i\Omega_0 u), & \text{for } u < 0, \end{cases} \quad (32)$$

as Ω_0 is the only root of $\mathfrak{D}_0(\omega)$ having negative imaginary part. For $-\varepsilon < u < 0$, both expressions in equation (32) can be used, even though the second is obviously preferable for computational reasons.

Some comments are necessary when $\gamma = 1$. In such case, the purely imaginary zero, Ω_1, vanishes, and the integral appearing in equation (28) is not defined; in fact, when absorption is absent, the number of particles of the system increases ceaselessly, and a real stationary solution to the Boltzmann equation does not exist. In fact, even in this case a stationary solution does exist for any finite energy, as particles tend to accumulate in time only at zero energy. From a mathematical point of view, such a case can be regarded as a limit situation in which $\gamma = 1 - \delta$, with $0 < \delta \ll 1$. A typical scattering density $\mathfrak{F}(u)$ obtained by means of solution (32) for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 10^{-4}$ is shown in Figure 2. In the figure, the analytical solution is compared with a numerical result obtained by resorting to Monte Carlo method. Due to the sufficiently small value of ζ, the agreement is excellent. However, if higher values of ζ are chosen, the approximate solution (32) is expected to fail and the full P_1 system (18) is to be solved.

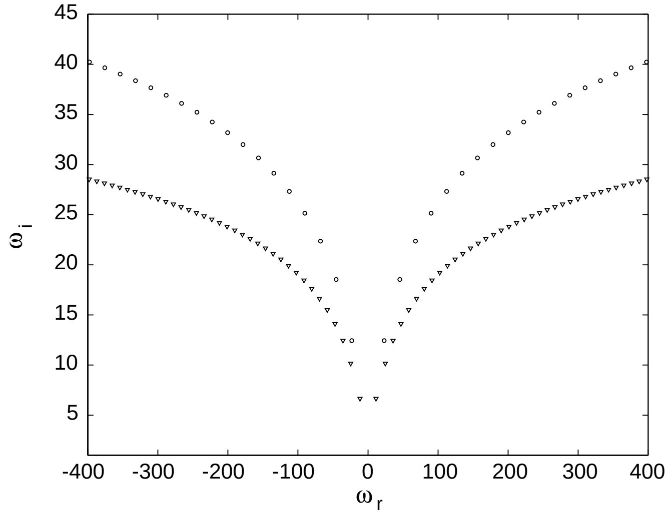

Fig. 3. Location of the zeroes of $\mathfrak{D}_0(\omega)$ (circles) and $\mathfrak{D}_1(\omega)$ (triangles), for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 5 \times 10^{-3}$.

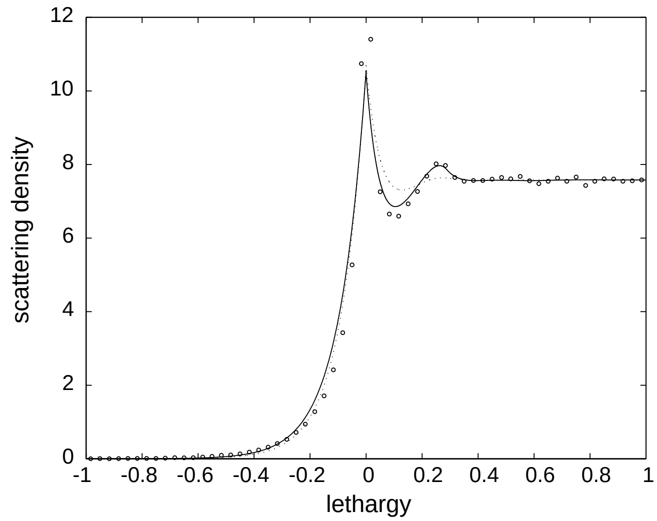

Fig. 4. Scattering density obtained by solving equation (20) (full line), equation (18) (dotted line) and by Monte Carlo simulation (circles), for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 5 \times 10^{-3}$.

5 Analytical solution of the complete P_1 model

The technique based on the Fourier transform, presented in the previous section, can also be employed to solve the complete P_1 system (18). In fact, noticing that the Fourier transform of the r.h.s. of the second equation (18) is

$$\frac{1}{(1-\alpha)(1-\sqrt{\alpha})}\left\{\frac{1-\exp\left(-(\frac{3}{2}+i\omega)\Delta u\right)}{\frac{3}{2}+i\omega} - \sqrt{\alpha}\frac{1-\exp\left(-(\frac{1}{2}+i\omega)\Delta u\right)}{\frac{1}{2}+i\omega}\right\}\widetilde{\mathfrak{J}}(\omega) \equiv \mathfrak{M}(\omega)\widetilde{\mathfrak{J}}(\omega), \quad (33)$$

the transform of the collision density can be expressed as in equation (26), simply substituting $\mathfrak{D}_0(\omega)$ with a new function, $\mathfrak{D}_1(\omega)$, defined as $\mathfrak{D}_0(\omega)$, except for $\langle\mu_0\rangle$, which is replaced by $\mathfrak{M}(\omega)$. Thus, the function $\mathfrak{D}_1(\omega)$ can be written as

$$\mathfrak{D}_1(\omega) = \frac{1}{\gamma} - \frac{1-\alpha\exp(-i\omega\Delta u)}{(1-\alpha)(1+i\omega)} - i\mathfrak{Z}(\omega)\omega(1+i\omega), \quad (34)$$

which has the same form as equation (27), having replaced ζ with by the function $\mathfrak{Z}(\omega)$, defined as

$$\mathfrak{Z}(\omega) = \zeta\frac{1-\gamma\langle\mu_0\rangle}{1-\gamma\mathfrak{M}(\omega)}\,. \quad (35)$$

In the limit $\omega \to 0$, $\mathfrak{Z}(\omega)$ coincides with ζ. The inverse transform, giving the scattering density $\mathfrak{F}(u)$, can be evaluated using again the residue theorem. A solution having the same form as equation (32) is obtained, the only difference being the value of the Ω_k and of the residues. In the appendix, an asymptotic formula for the Ω_k, similar to equation (29), is deduced. In Figure 3, the location of the zeroes of $\mathfrak{D}_1(\omega)$ having a non-vanishing real part, for $\gamma = 1$, $\alpha = 0.75$ and $\zeta = 5 \times 10^{-3}$ is shown, together with the corresponding zeroes of $\mathfrak{D}_0(\omega)$. The corresponding scattering densities are presented in Figure 4; the solution obtained with the Monte Carlo method is reported also. Having chosen a value of ζ higher than the one used for the case reported in Figure 2, the complete P_1 model provides a far better solution than the one obtained by the simplified P_1 model. If a higher value of ζ is chosen, also the complete P_1 model begins to fail, as the electric field cannot be considered weak. In such cases, the P_1 approximation is not sufficient to obtain an accurate solution and more terms must be included in the spherical harmonics expansion [1,10].

Appendix: Properties of the zeroes of $\mathfrak{D}_0(\omega)$ and of $\mathfrak{D}_1(\omega)$

By introducing the new complex variable $z = 1 + i\omega = x + iy$, the equation $\mathfrak{D}_0(\omega) = 0$ can be written as

$$\frac{\gamma}{1-\alpha}\left(1-\exp(-z\Delta u)\right) = z - \zeta\gamma\left(z^3 - z^2\right)\,. \quad (A.1)$$

As equation (A.1) has only real coefficients, if z is one of its roots, also its conjugate z^* is a solution. For that reason, in the following only the roots with $y > 0$ are considered. By separating real and imaginary part of equation (A.1), a system of equations for x and y is obtained:

$$\begin{cases} \dfrac{\gamma}{1-\alpha}\exp(-x\Delta u)\cos(y\Delta u) = \\ \qquad \dfrac{\gamma}{1-\alpha} - x + \zeta\gamma\,\mathrm{Re}\left(z^3 - z^2\right)\,, \\ \dfrac{\gamma}{1-\alpha}\exp(-x\Delta u)\sin(y\Delta u) = \\ \qquad y - \zeta\gamma\,\mathrm{Im}\left(z^3 - z^2\right)\,. \end{cases} \quad (A.2)$$

By taking the square of both equations and summing, one obtains

$$\left(\frac{\gamma}{1-\alpha}\right)^2 \exp(-2x\Delta u) = \left[p_0(x) + p_1(x)y^2\right]^2 + y^2\left[p_2(x) + \zeta\gamma y^2\right]^2 , \qquad \text{(A.3)}$$

being

$$\begin{aligned} p_0(x) &= \frac{\gamma}{1-\alpha} - x + \zeta\gamma(x^3 - x^2) , \\ p_1(x) &= \zeta\gamma(1-3x) , \\ p_2(x) &= 1 + \zeta\gamma(2x - 3x^2) . \end{aligned} \qquad \text{(A.4)}$$

Equation (A.3) can be regarded as a third-order algebraic equation for y^2, whose coefficients are functions of x. Only one root of equation (A.3) is real; such solution provides an explicit relationship connecting x and y, from which the locus of the zeroes of $\mathfrak{D}_0(\omega)$ in the ω plane can be readily drawn.

From system (A.2), an asymptotic expression for the solutions of the equation $\mathfrak{D}_0(\omega) = 0$ can be obtained. Considering that $|x| \ll |y|$, the term $z^3 - z^2$ in both equations can be approximated as $-3xy^2 - iy^3$. If the first of equations (A.2) is divided by the second, one obtains

$$\cot(y\Delta u) \sim \frac{\dfrac{\gamma}{1-\alpha} - x\left(1 + 3\zeta\gamma y^2\right)}{y\left(1 + \zeta\gamma y^2\right)} . \qquad \text{(A.5)}$$

When $y \to \infty$, the r.h.s. tends to zero. This means that, asymptotically, $y\Delta u \sim \frac{\pi}{2} + k\pi$, and, consequently, sin $(y\Delta u) \sim (-1)^k$ in the second of equations (A.2). When positive values of y are considered, the r.h.s. of equation (A.2) is positive; this requires that $k = 2n$. In conclusions, the imaginary part of the solutions of equation (A.1) are given by the following asymptotic expression:

$$y \sim \frac{1}{\Delta u}\left(\frac{\pi}{2} + 2n\pi\right) . \qquad \text{(A.6)}$$

The corresponding real part of the roots can be evaluated by considering the second equation (A.2), having approximated Im $\left(z^3 - z^2\right)$ as $-y^3$ and having used the asymptotic expression (A.6) for y. One obtains

$$x \sim -\frac{1}{\Delta u}\log\left\{\frac{1-\alpha}{\gamma\Delta u}\left(\frac{\pi}{2} + 2n\pi\right) \times \left[1 + \frac{\zeta\gamma}{\Delta u^2}\left(\frac{\pi}{2} + 2n\pi\right)^2\right]\right\} . \qquad \text{(A.7)}$$

In a similar way, the asymptotic behaviour of the zeroes of $\mathfrak{D}_1(\omega)$ can be deduced. In fact, the equation $\mathfrak{D}_1(\omega) = 0$ can be written as

$$\frac{\gamma}{1-\alpha}\left(1 - \exp(-z\Delta u)\right) = z - \frac{\widehat{\zeta}\gamma\left(z^3 - z^2\right)}{\Psi(z)} , \qquad \text{(A.8)}$$

being $\widehat{\zeta} = (1 - \langle\mu_0\rangle)\,\zeta$, and having introduced the function $\Psi(z)$, defined as

$$\Psi(z) = 1 - \frac{\gamma}{(1-\alpha)\left(1-\sqrt{\alpha}\right)} \times \left(\frac{1 - \sqrt{\alpha}\exp(-z\Delta u)}{z + \frac{1}{2}} - \frac{\sqrt{\alpha} - \exp(-z\Delta u)}{z - \frac{1}{2}}\right) . \qquad \text{(A.9)}$$

For large z, $\Psi(z)$ behaves asymptotically as

$$\Psi(z) \sim 1 - \frac{\gamma}{1-\alpha}\frac{\exp(-z\Delta u)}{z} , \qquad \text{(A.10)}$$

and equation (A.8) can be approximated by the following equation:

$$\frac{\gamma}{1-\alpha} = \mathfrak{L} + z + \frac{\widehat{\zeta}\gamma z^4}{\mathfrak{L} - z} , \quad \mathfrak{L} = \frac{\gamma}{1-\alpha}\exp(-z\Delta u), \qquad \text{(A.11)}$$

which can be solved with respect to $\mathfrak{L}$, obtaining

$$\mathfrak{L}^2 \sim z^2 - \widehat{\zeta}\gamma z^4 . \qquad \text{(A.12)}$$

Assuming again that $|x| \ll |y|$, the r.h.s. of equation (A.12) can be approximated as $-y^2 - \widehat{\zeta}\gamma y^4$. Being

$$\mathfrak{L}^2 = \left(\frac{\gamma}{1-\alpha}\right)^2 \exp(-2x\Delta u - 2iy\Delta u) , \qquad \text{(A.13)}$$

this requires that $2y\Delta u \sim k\pi$. Thus the real part of equation (A.12) can be written as

$$(-1)^k\left(\frac{\gamma}{1-\alpha}\right)^2 \exp(-2x\Delta u) \sim -y^2\left(1 + \widehat{\zeta}\gamma y^2\right) . \qquad \text{(A.14)}$$

Equation (A.14) requires k to be odd. In conclusion, the asymptotic behaviour of the zeroes of $\mathfrak{D}_1(\omega)$ are given by

$$\begin{aligned} y &\sim \frac{1}{\Delta u}\left(\frac{\pi}{2} + n\pi\right) , \\ x &\sim -\frac{1}{\Delta u}\log\left\{\frac{1-\alpha}{\gamma\Delta u}\left(\frac{\pi}{2} + n\pi\right) \times \left[1 + \frac{\hat{\zeta}\gamma}{\Delta u^2}\left(\frac{\pi}{2} + n\pi\right)^2\right]^{\frac{1}{2}}\right\} . \end{aligned} \qquad \text{(A.15)}$$

In the derivation, the term y^2 has been retained together with $\widehat{\zeta}\gamma y^4$, in order to provide a good approximation even when $\widehat{\zeta}\gamma y^2 \ll 1$.

References

1. S. Shankar, K.F. Jensen, IEEE Trans. Plasma Sci. **23**, 780 (1995).
2. L.C. Pitchford, S.V. O'Neil, J.R. Rumble, Phys. Rev. A **23**, 294 (1981).
3. L.C. Pitchford, A.V. Phelps, Phys. Rev. A **25**, 540 (1982).

4. S. Yachi, Y. Kitamura, K. Kitamori, H. Tagashira, J. Phys. D **21**, 914 (1988).
5. S. Yachi, H. Dates, K. Kitamori, H. Tagashira, J. Phys. D **24**, 573 (1991).
6. M.C. Vecchi, M. Rudan, IEEE Trans. Electron. Dev. **45**, 230 (1998).
7. S. Reggiani, M.C. Vecchi, M. Rudan, IEEE Trans. Electron. Dev. **45**, 2010 (1998).
8. M.C. Vecchi J. Mohring, M. Rudan, VLSI Design **6**, 239 (1998).
9. A. Greiner, M.C. Vecchi, M. Rudan, Semicond. Sci. Technol. **13**, 1080 (1998).
10. D. Schroeder, D. Ventura, A. Gnudi, G. Baccarani, Electron. Lett. **28**, 995 (1992).
11. W.L. Morgan, B.M. Penetrante, Comput. Phys. Commun. **58**, 127 (1990).
12. G. Colonna, C. Gorse, M. Capitelli, R. Winkler, J. Wilhelm, Chem. Phys. Lett. **213**, 5 (1993).
13. R. Brunetti, M.C. Vecchi, M. Rudan, VLSI Design **6**, 161 (1998).
14. D. Ventura, A. Gnudi, G. Baccarani, Riv. Nuovo Cimento **18**, No. 6 (1995).
15. I. Armenise, M. Capitelli, R. Celiberto, G. Colonna, C. Gorse, A. Laganà, Chem. Phys. Lett. **227**, 157 (1994).
16. G. Capriati, G. Colonna, C. Gorse, M. Capitelli, Plasma Chem. Plasma Process. **12**, 237 (1991).
17. G.G.M. Coppa, *Effetto dello Scattering sulla Distribuzione di Velocità di Particelle Cariche in un Campo Elettrico: Teorema di Equivalenza*, PT-DE-IN-281, Politecnico di Torino, 1992.
18. K.M. Case, F. de Hoffmann, G. Placzek, *Introduction to the Theory of Neutron Diffusion* (Los Alamos Scientific Laboratory, 1953).
19. Yu. Raizer, *Gas Discharge Physics* (Springer, Berlin, 1993).
20. R.V. Meghreblian, D.K. Holmes, *Reactor Analysis* (McGraw-Hill, New York, 1960).